

Geometrical characterization of non-Markovianity

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We introduce a new tool for the quantitative characterisation of the departure from Markovianity of a given dynamical process. Our tool can be applied to a generic N -level system and extended straightforwardly to Gaussian continuous-variable systems. It is linked to the change of the volume of physical states that are dynamically accessible to a system and provides qualitative expectations in agreement with some of the analogous tools proposed so far. We illustrate its predictive power by tackling a few canonical examples.

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The interaction with an environment leads a quantum system to dissipate energy and lose its coherence. The process, however, needs not be monotonic and the system may temporarily recover some of the lost energy and/or information. This is the essence of a non-Markovian behavior, which can be characterized and quantified in many different ways [1, 2]. One possibility is to look for temporary increases of the entanglement shared by the system with an isolated ancilla, which amounts to measure the deviation from divisibility of the dynamical map describing the system's reduced evolution [3] (RHP measure). A different approach [4, 5] relies on measuring the distinguishability of two *optimal* initial states that have evolved through the same quantum channel, looking for any non-monotonicity (BLP measure). Further proposals have been put forward, based on the decay rates entering the master equation [6], on the Fisher information flow [7], on the use of the quantum mutual information [8] or of channel capacities [9], and on spectral considerations [10]. This variety of tools highlight the multi-faceted nature of the problem embodied by a characterisation of non-Markovian dynamics and its inherent difficulty, which prevents the formulation of a unique tool.

In this paper, we contribute to the quest for sharp tools able to capture the various aspects with which non-Markovianity manifests itself and propose a method that qualifies non-Markovian evolutions based on the rate of change of the volume of accessible states of the evolved system. As for the divisibility-based approach [3], this is a characteristic of the map itself that does not depend on the initial state(s) of the system (nor needs to be optimized over them). A quantum evolution is Markovian if it is an element of any one-parameter continuous completely positive semigroup: in this case the process is unidirectional and there is no recovery of energy/information/coherence by the system. This implies that the domain's volume of the dynamical map

decreases monotonically. On the contrary, we associate non-Markovianity of the dynamics to a growth of this domain's volume. We thus define a quantifier of non-Markovianity as the sum of the (temporary) volume increases which occur during the time evolution.

In the case of a single qubit, this can be linked to the BLP measure [4], as the trace distance coincides with the Euclidean distance on the Bloch sphere and the pair of states that maximize the measure lie on the boundary of the convex subspace of physical states [11]. As a result, if the trace distance decreases monotonically, so does the volume, which is however much easier to evaluate through the determinant of the dynamical map, as we address in details in this paper. Our aim here is to formalize the intuition at the basis of our proposal towards the construction of a new tool for the quantitative characterisation of non-Markovianity. It should be stressed that ours is not *yet another attempt at the quantification of the degree of non-Markovianity* of a given dynamical process, but the proposal for a novel way to reveal effects of an evolution departing from the features of Markovianity that went so far overlooked. Our proposal enjoys features of practicality and intuition of interpretation that are somehow missing from analogous, otherwise equally valid quantifiers.

Systems with finite dimensional Hilbert spaces. Irrespective of the initial open system state, a reduced time evolution derived from the unitary dynamics of a larger system can always be described by a linear, Hermitian map [12], not necessarily completely positive due, e.g., to the presence of initial system-environment correlations [13]. A Markovian or memory-less behavior leads to master equations in the Lindblad form [14, 15], with the map obeying the semigroup composition law. We consider a positive trace preserving map

$$\phi_t : \hat{\rho}(0) \rightarrow \hat{\rho}(t) = \phi_t[\hat{\rho}(0)] \quad (1)$$

for the quantum state of a N -level open system, which can be expressed through a generalized Bloch vector \mathbf{r} , whose components are the expectation values of the traceless, hermitian generators of $SU(N)$, $G_i (i = 1, \dots, N^2 - 1)$, for which $\text{Tr}[\hat{G}_i \hat{G}_j] = \delta_{ij}$. By including the identity $\hat{G}_0 = \mathbb{I}/\sqrt{N}$, any state can be written as

$$\hat{\rho} = \sum_{\alpha=0}^{N^2-1} \text{Tr}[\hat{\rho} \hat{G}_\alpha] \hat{G}_\alpha \equiv \sum_{\alpha=0}^{N^2-1} r_\alpha \hat{G}_\alpha \quad (2)$$

with $\vec{r} = (1/\sqrt{N}, \mathbf{r})$. A systematic construction of the $\{\hat{G}_\alpha\}$ is given in Refs. [16, 17] and leads us to $\{\hat{G}_\alpha\}_{\alpha=1}^{N^2-1} = \{\hat{u}_{jk}, \hat{v}_{jk}, \hat{w}_l\}/\sqrt{2}$ with

$$\begin{aligned} \hat{u}_{jk} &= |j\rangle \langle k| + |k\rangle \langle j|, & \hat{v}_{jk} &= -i(|j\rangle \langle k| - |k\rangle \langle j|), \\ \hat{w}_l &= \sqrt{\frac{2}{l(l+1)}} \sum_{j=1}^l (|j\rangle \langle j| - l|l+1\rangle \langle l+1|), \end{aligned} \quad (3)$$

where the span of the indices is such that $1 \leq j < k \leq N$, $1 \leq l \leq N-1$ and $\{|m\rangle\}_{m=1}^N$ is an orthonormal basis of the open system's Hilbert space. This gives Pauli spin operators for $N=2$ and Gell-Mann operators for $N=3$.

Writing the map in Eq. (1) in this basis, one gets

$$\vec{r}_t = \mathbf{F}(t) \vec{r}_0, \quad \text{with } F_{\alpha\beta}(t) = \text{Tr}[\hat{G}_\alpha \phi_t[\hat{G}_\beta]]. \quad (4)$$

As $F_{0\beta}(t) = \delta_{0\beta}$, this is an affine transformation for the Bloch vector. Letting $q_\beta(t) = F_{\beta 0}$, we have

$$\mathbf{F}(t) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{q}_t & \mathbf{A}_t \end{pmatrix} \rightarrow \mathbf{r}_t = \mathbf{A}_t \mathbf{r}_0 + \mathbf{q}_t/\sqrt{N}. \quad (5)$$

The real matrix \mathbf{A}_t can be decomposed as $\mathbf{A}_t = \mathcal{O}_t^1 \mathbf{D}_t \mathcal{O}_t^{2T}$, where \mathcal{O}_t^n 's are orthogonal matrices and \mathbf{D} is a positive semi-definite diagonal one. In what follows, we will indicate with $|M|$ the determinant of a matrix M . The findings above imply that $|\mathbf{F}_t| = |\mathbf{A}_t| = |\mathbf{D}_t|$. The action of \mathbf{F} is given by a first rotation (possibly composed with an inversion), then a shrink of the Bloch vector followed by a final rotation plus a translation. Its determinant gives the contraction factor for the volume of accessible states, given by the measure of the set of evolved Bloch vectors, with respect to its value at $t=0$.

The set of physical Bloch-vectors for an N -level system is given by [18]

$$B_N = \{\mathbf{r} \in \mathbb{R}^{N^2-1} : (-1)^j a_j(\mathbf{r}) \geq 0 \ (j = 1, \dots, N)\},$$

where $a_j(\mathbf{r})$ are the coefficients of the characteristic polynomial $\det(x\mathbb{I}^N - \hat{\rho})$ with $\hat{\rho} = \frac{1}{N}\mathbb{I}^{(N)} + \sum_{i=1}^{N^2-1} r_i \hat{G}_i$. In spherical coordinates, the volume element of B_N is

$$d^N V = \left\| \frac{\partial(r_i)}{\partial(R, \phi_j)} \right\| dR d\phi_1 d\phi_2 \cdots d\phi_{n-1} \quad (6)$$

and, by direct substitution, it is straightforward to check that any positive trace-preserving map described by Eq. (4) induces the change

$$d^N V(t) = \|\mathbf{A}_t\| d^N V(0). \quad (7)$$

Therefore, $\|\mathbf{A}_t\|$ describes the change in volume of the set of states accessible through the evolution of the reduced state. In particular, $\|\mathbf{A}_t\|$ decreases monotonically for any positive, linear and trace preserving map [19], and so it does for any element of a completely positive continuous one-parameter semigroup. Indeed, if $\phi_t = e^{t\mathcal{L}}$ with

$$\mathcal{L} \hat{\rho} = i[\hat{\rho}, H] + \sum_{\alpha, \beta} \gamma_{\alpha, \beta} \left(C_\alpha \hat{\rho} C_\beta^\dagger - \frac{1}{2} \{C_\beta^\dagger C_\alpha, \hat{\rho}\} \right) \quad (8)$$

$\gamma \geq 0$, and H the Hamiltonian of the system, we get $|\mathbf{A}_t| = e^{-N \text{Tr}[\gamma]}$, which is a constant.

Time dependent generators of the form in Eq. (8) with $\gamma(t) \geq 0$ lead to time-dependent Markovian processes. The dynamical map $\phi_{t+\tau, t} = \exp[T \int_t^{t+\tau} \mathcal{L} dt]$, although not being part of a dynamical semigroup, is divisible and can be written as the composition of two CPT maps

$$\phi_{t+\tau, 0} = \phi_{t+\tau, t} \phi_{t, 0} \quad (\forall \tau, t \geq 0). \quad (9)$$

As a consequence, in this case too the determinant is monotonically decreasing. These considerations lead us to define a new way to quantify the non-Markovian character of a quantum evolution through the variation of the volume of accessible states

$$\mathcal{N}_V = \frac{1}{V(0)} \int_{\frac{dV(t)}{dt} > 0} \frac{dV(t)}{dt} = \int_{\frac{d\|\mathbf{F}_t\|}{dt} > 0} \frac{d\|\mathbf{F}_t\|}{dt} \quad (10)$$

The intuitive meaning of this definition is illustrated in Fig. 1, where the time evolution of the determinant is explicitly shown for the case of a two-level atom spontaneously decaying in a structured environment (cf. *Example 1* for details). The monotonous volume decay characterizing a Markovian time evolution is contrasted with a non-Markovian behavior in which a non-zero accessible volume re-appear after being fully decayed.

Besides the geometric interpretation, such a measure has a simple physical meaning based on the change of the classical information encoded in the states. Suppose that a set of quantum states is given, whose elements are characterized by an arbitrary distribution of Bloch vectors within B_N , described by a probability density $p(\mathbf{r})$. The corresponding differential entropy is

$$h[p(\mathbf{r})] = - \int_{B_N} p(\mathbf{r}) \log_2 p(\mathbf{r}) dV_N. \quad (11)$$

If such states are taken as the initializations of the map ϕ_t , after a time t the probability density function is rescaled as $p'(\mathbf{r}_t) = p(\mathbf{r}_t)/\|\mathbf{A}_t\|$ and the entropy changes accordingly as

$$h[p'(\mathbf{r}_t)] - h[p(\mathbf{r}_0)] = \log_2 \|\mathbf{A}_t\|. \quad (12)$$

Thus, a contraction of volume is equivalent to a loss of classical information.

The BLP measure also enjoys an information theoretical interpretation, but a comparison between the two quantifiers is difficult. Indeed, although it is known that the optimal states that enter the measure proposed in Ref. [4] lie on the boundary of the volume accessible throughout the dynamics [11], this does not necessarily imply a connection with the measure of such volume. On the other hand, the difference between \mathcal{N}_V and the RHP measure is simpler to describe: as the determinant is contractive under composition of positive maps, it follows that it does not increase whenever the intermediate map $\phi_{t+\tau,t}$ in Eq. (9) is non positive. The entanglement-based measure of Ref. [3], on the other hand, is non-zero in the less restrictive condition of the map being not completely positive. Therefore, if $\phi_{t+\tau,t}$ is positive but not completely positive, the map is non-divisible. Nonetheless, we find $\mathcal{N}_V = 0$.

From a practical viewpoint, the experimental evaluation of \mathcal{N}_V passes through the determination of the volume of the set of evolved states found from the contraction of B_N . This is an ellipsoid for the case of a qubit. As the number of evolved states that any realistic experimental implementation can sample is finite, it would be a precious piece of information to know which are the best initial states to use in order to determine the set of physically accessible states at a given time t of the dynamics and its volume. For this purpose, let us consider N initial Bloch vectors, evolved up to time t and arranged as the columns of a matrix \mathbf{P}_t . We have from Eq. (5), that such vector evolves as $\mathbf{P}_t = \mathbf{A}_t \mathbf{P}_0 + \mathbf{Q}_t$, where the columns of \mathbf{Q}_t are given by the \mathbf{q}_t 's (which provides the form of the evolved state for a maximally mixed initial condition). From this it follows that $|(\mathbf{P}_t - \mathbf{B}_t)(\mathbf{P}_t - \mathbf{B}_t)^T| = (|\mathbf{A}_t|)^2(|\mathbf{P}_0|)^2$. In turn, if we choose as initial Bloch vectors the elements of any orthogonal basis in \mathbb{R}^{N^2-1} plus the null vector corresponding to the maximally mixed state, then their time evolutions (arranged to form the matrix $\mathbf{P}_t - \mathbf{Q}_t$) gives the determinant of the map, from which the measure \mathcal{N}_V easily follows. Therefore, the geometric measure of non-Markovianity in Eq. (10) can be revealed experimentally by performing a state tomography at different times for $N^2 - 1$ initial orthogonal states. This will be sufficient to evaluate the change in volume of the accessible states without the need for prior knowledge about the environment or the coupling. For the case of a qubit, this is illustrated in Fig. 2 (a), where the three initial Bloch vectors corresponding to the canonical basis of \mathbb{R}^3 are shown to evolve into the extreme points on the semi-axis of the ellipsoid that comprises all the possible accessible states of the evolution.

Systems with infinite dimensional Hilbert space. We can extend our idea to the less intuitive context of continuous variable systems, in which the Hilbert space is in-

finite dimensional and it is not possible to describe a state through a finite number of parameters. However, the restriction to Gaussian state and Gaussian-preserving processes helps overcoming this issue.

We consider a system made of n bosonic modes $k = 1, \dots, n$, each described by the annihilation and creation operators \hat{a}_k and \hat{a}_k^\dagger [corresponding position and momentum operators $\hat{q}_k = 1/\sqrt{2}(\hat{a}_k + \hat{a}_k^\dagger)$ and $\hat{p}_k = -i/\sqrt{2}(\hat{a}_k - \hat{a}_k^\dagger)$, respectively]. Defining the vector of operators $\hat{\mathbf{R}} = (\hat{q}_1, \hat{p}_1, \dots, \hat{q}_n, \hat{p}_n)^T$, the commutation relations can be written as $[\hat{R}_k, \hat{R}_l] = i\Omega_{kl}$ where Ω_{kl} are elements of the symplectic matrix $\mathbf{\Omega} = \bigoplus_{k=1}^n \omega$ with $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. A Gaussian state is completely characterized by its first and second statistical moments, given respectively by $\langle \mathbf{R} \rangle$ and the covariance matrix σ defined as

$$\sigma_{kl} = \frac{1}{2} \langle \{\hat{R}_k, \hat{R}_l\} \rangle - \langle \hat{R}_k \rangle \langle \hat{R}_l \rangle. \quad (13)$$

First moments can be adjusted to be null by local unitary operations. It can be shown that any evolution resulting from the reduction of a symplectic evolution on a larger Hilbert space can be described, in terms of second moments, by the equation

$$\sigma_t \rightarrow X_t^T \sigma_0 X_t + Y_t \quad (14)$$

where X and Y are $2n \times 2n$ real matrices fulfilling the relation $Y + i\mathbf{\Omega} - iX^T \mathbf{\Omega} X \geq 0$. Viceversa, any evolution of this kind may be interpreted as the reduction of a larger symplectic evolution [20]. In full analogy with what we did in (4), by choosing a basis $\{G_j\}$ for the space of $2n \times 2n$ matrices, we can write such map as

$$\sigma_t = \sum_{jk} \text{Tr}[X_t^T G_k X_t G_j] \text{Tr}[\sigma_0 G_k] G_j + \text{Tr}[Y_t G_j] G_j. \quad (15)$$

Eq. (15) can be recast as the \mathbb{R}^{4N^2} affine transformation $\mathbf{s}_0 \rightarrow \mathbf{s}_t = \mathbf{X}_t \mathbf{s}_0 + \mathbf{Y}_t$. We then define a measure of non-Markovianity in a way fully analogous to what has been done above for the case of a discrete-variable system, *i.e.* as in Eq. (10) with the replacement $\mathbf{F}_t \rightarrow \mathbf{X}_t$.

If the evolution is unitary, the associated transformation has a constant determinant, equal to one. For a single mode in a generic noisy Markovian channel, we have [21]

$$\sigma_t = e^{-\Gamma t} \sigma_0 + (1 - e^{-\Gamma t}) \sigma_\infty, \quad (16)$$

which gives $|\mathbf{X}_t| = e^{-4\Gamma t}$. Therefore, every increase in $|\mathbf{X}_t|$ signals non-Markovianity.

Having introduced our formalism, we now illustrate our proposal with the aid of a few significant examples.
Example 1: Spontaneous emission into a leaky cavity. Consider a single two-level atom with transition frequency ω_0 interacting with a vacuum electromagnetic

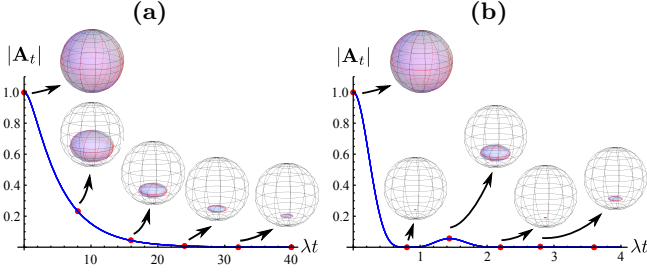


FIG. 1: (Color online) Time evolution of the determinant of the map for a generic Markovian [panel (a)] and non-Markovian dynamics [panel (b)]. Quantitatively, the curves displayed in the figure correspond to the spontaneous emission of a two-level system in a resonant leaky cavity (cf. *Example 1*) with the Lorentzian spectral density given in Eq. (17). Panel (a) shows the (Markovian) case with $\gamma_0/\lambda = 0.1$ (bad cavity limit), while panel (b) shows a (non-Markovian) evolution in the good cavity limit with $\gamma_0/\lambda = 10$. We also picture the set of accessible states, whose volume changes in time according to the behavior of the determinant.

field having a Lorentzian spectral density (mimicking a leaky cavity) [5]. Taking

$$J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{(\omega_0 - \Delta - \omega)^2 + \lambda^2}, \quad (17)$$

where Δ is the detuning between the atomic and the cavity frequency, the atomic state at time t reads

$$\hat{\rho}_A(t) = \begin{pmatrix} |\Gamma(t)|^2 \rho_0^{++} & \Gamma(t) \rho_0^{+-} \\ \Gamma(t)^* \rho_0^{-+} & (1 - |\Gamma(t)|^2) \rho_0^{--} + \rho_0^{--} \end{pmatrix} \quad (18)$$

with $\Omega_{\pm} = \Delta - i\lambda \pm \sqrt{(\Delta - i\lambda)^2 + 2\gamma_0\lambda}$

$$\Gamma(t) = \frac{e^{-\frac{it\Omega_-}{2}} \Omega_+ - e^{-\frac{it\Omega_+}{2}} \Omega_-}{2(\Omega_+ - \Delta + i\lambda)}. \quad (19)$$

The evolution of the Bloch vector is ruled by

$$\mathbf{A}_t = \begin{pmatrix} \Re \Gamma(t) & \Im \Gamma(t) & 0 \\ -\Im \Gamma(t) & \Re \Gamma(t) & 0 \\ 0 & 0 & |\Gamma(t)|^2 \end{pmatrix}, \quad (20)$$

whose determinant, $|\mathbf{A}_t| = |\Gamma(t)|^4$, is shown in Fig. 1 against the dimensionless time λt . The corresponding non-Markovianity measure \mathcal{N}_V is reported in Fig. 2, from which it is clear that a strongly non-Markovian behavior is found for a resonant coupling and invoking the so-called good-cavity limit $\gamma_0 \gg \lambda$. A similar result is obtained with the RHP measure, [3] which is given by the integral of $(1/2)\Re[\partial_t \ln \Gamma(t)]$. This is in agreement with BLP as well, which turns out to depend on $|\Gamma(t)|$ [5].

Example 2: Pure dephasing. Let us consider a qubit undergoing a purely dephasing dynamics, expressed in terms of a decoherence factor $\nu(t)$ as

$$\phi_t^{(d)}[\rho(0)] = \begin{pmatrix} \rho^{++} & \nu(t) \rho^{+-} \\ \nu(t) \rho^{-+} & \rho^{--} \end{pmatrix}. \quad (21)$$

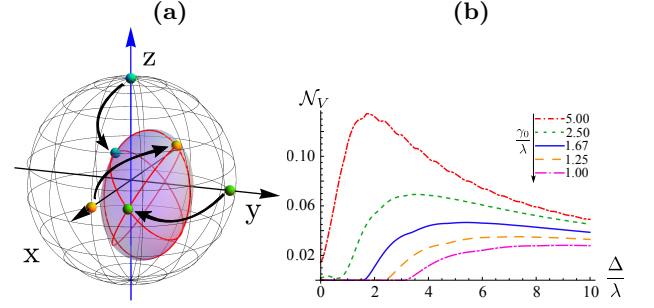


FIG. 2: (Color online) Panel (a) Example of an ellipsoid representing the image of a random map for a qubit. The canonical basis of \mathbb{R}^3 is mapped onto the corresponding colored circles highlighted on the ellipsoid, from which the volume can be obtained. Panel (b) shows the non-Markovianity measure \mathcal{N}_V as a function of the detuning, for the spontaneous emission dynamics into a reservoir with Lorentzian spectral density (cf. *Example 1*).

In this case, the BLP and RHP measures coincide [22, 23]. In fact, the trace distance optimised over the initial qubit preparation gives us $D[\rho_1^{opt}, \rho_2^{opt}] = |\nu(t)|$. In turn, $|\nu(t)|$ is exactly the value of the concurrence between a system qubit and ancilla, initially prepared in a maximally entangled state and undergoing a unilateral dephasing mechanism. As for our proposal, the evolution of the Bloch vector is determined by the matrix

$$\mathbf{A}_t = \begin{pmatrix} \Re \nu(t) & \Im \nu(t) & 0 \\ -\Im \nu(t) & \Re \nu(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (22)$$

which has $\|\mathbf{A}_t\| = |\nu(t)|^2$. The geometric measure \mathcal{N}_V thus gives the same behavior predicted by the other two quantifiers.

We have proposed a geometrically motivated quantifiers of non-Markovianity that is explicitly linked to the variations in the volume of the physical states dynamically accessible by a given open system. From an information theoretical perspective, such measure, which provides predictions that are, in general, in qualitative agreement with those coming from some of the most popular tools for the characterisation of non-Markovianity proposed to date, is linked to the loss/regain of classical information over the evolving system. We have shown how an estimate of the proposed measure is possible through only a polynomial number of measures, while the proposed formal quantifier itself enjoys a straightforward extension to the Gaussian continuous-variable scenario. We have illustrated our proposal through a series of examples. We hope that the appealing aspects of practicality and intuitive nature of our proposal will soon spur the attention of the community interested in open-system dynamics.

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